



Actuarial Pricing Mechanisms Under Advanced Demographic Mortality Rate Intensity Function Involving Analytic Heligman-Pollard's Third Law

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Abstract

The objectives of this study are to derive actuarial present values of whole life annuity, whole life insurance and net level premiums. The net premium policy value on a scheme in force within period $\zeta > 0$ years after it has been incepted represents the expected value at the time of the net future loss random variable that does not incorporate any allowance for expenses. The net premiums estimated on the policy value basis are obtained using the equivalence principle and not the actual premium paid. Computational evidence from the results reveals that the whole life annuity-due is the discounted survival state, showing the insured's remaining potential to finance mortality risk through time. The whole life insurance value is the price of death claim where value increases as mortality increases and the time horizon declines. At age 0, $P_0^{(M)} = 0.00126613$ while $P_0^{(F)} = 0.00085503$ and small magnitude of the premium is because the probability of death is very low, expected claim cost is minimal and payments are spread over a long premium-paying period. The male premium is higher than the female premium because of higher mortality rate intensities and hence $P_x^{(M)} > P_x^{(F)}$. Therefore, this reflects pure mortality-driven pricing under the equivalence principle. The premium monotonically increases for both sexes at age 40 as $P_{40}^{(M)} = 0.00959298$ while $P_{40}^{(F)} = 0.00641200$ while at age 60, $P_{60}^{(M)} = 0.02635719$ and $P_{60}^{(F)} = 0.01730709$. At age 100 $P_{100}^{(M)} = 0.26592026$ and $P_{100}^{(F)} = 0.18397285$. This increase is actuarially consistent with the equivalence principle. The function A_x increases because of higher probability of death and $\ddot{a}_x$ decreases as a result of shorter premium-paying period. Hence the ratio grows rapidly at advanced ages.

Keywords: Random loss, net premium, life annuity, life insurance, policy.

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Introduction

Parsimonious mortality functions occupy key positions in actuarial science and demography by giving analytically tractable formulations of age-specific death rates. Within these laws, the Heligman-Pollard mortality function developed in Heligman and Pollard (1980) remains one of the most analytic models for incorporating the full age-specific mortality trend. The law expresses the odds of death at

defined age x as aggregate of three auxiliary mortality components associate with early childhood, young adulthood and older ages. Since its evolution in mortality, a few papers have applied of the Heligman-Pollard model in some actuarial frameworks. Its parameters have been used to demonstrate mortality patterns in life tables constructions so as to contrast gender mortality and to investigate long-term variations in mortality trends under

changing epidemiological conditions. For example, the function was used to investigate the consequence of epidemics like HIV on mortality age trends associating variations in parameter values to epidemic intensity and adult mortality shifts, focusing its ability on demographic interpretation.

Extant literatures have expanded the law using Bayesian and dynamic frameworks to address challenges in projecting and quantifying parameter uncertainty. Bayesian methods under latent Gaussian formulations have been constructed to forecast mortality rates jointly with uncertainty assessments, revealing both the contributions and the problems of dynamic generalisations of the Heligman-Pollard's law. Furthermore, recent software implementations such as the *BayesMortalityPlus* package combine Heligman-Pollard's based life table generation with Bayesian tools on mortality graduation and projection to integrate stochastic mortality domain and other distributional assumptions into model computation.

Thomas (2019) discussed accounted for a detailed treatment of demography focusing on mortality analysis, population predictions and numerical implications in policy formulation. The core contribution is the combination of theoretical demographic analysis and its potential applications in actuarial life valuations and pension planning. The author gives comprehensive arguments of life table indicators and decomposition methods that make the theoretical foundations for life table estimation and smoothing. It is essentially useful in differentiating among crude rates, age-specific rates and standardised measures for constructing and interpreting mortality tables. Nevertheless, it does not explain fully on advanced stochastic mortality modeling methods like Lee-Carter or Bayesian techniques. Although it gives solid background, the discussion of parametric mortality laws such as Heligman-Pollard and current projection methods obviously lack depth.

The BPJS Kesehatan (2023) life table is an essential application to the Indonesian actuarial practice. Its core contribution covers offering current, nationally important mortality data to

mirror insured's population experience. The work supplies basic life table functions q_x and l_x which are critical for pricing and reserving. The dataset is essentially useful for estimating health-associated mortality risks so as to determine the financial health of national health insurance policies. The weakness of the paper is rooted in its discussion as a demographic table rather than an actuarial derivation technique. The work dwells much on mortality rate presentation and summary measures with sparse mortality data analysis process or graduation techniques. It has problem of selection bias because insured's population does not exactly reflect the general population. Moreover, without deep explanation on graduation methodologies, it offers restricted explanation for technical novelty in mortality estimations.

In AAJI (2019), the Indonesian life table IV is a key stone in life insurance pricing for Indonesia. Its main contribution to knowledge centers on the construction of mortality benchmark for life offices estimated from life office experience data. The table proves to be efficient in underwriting, reserving and solvency appraisal of life insurers. By integrating current mortality pattern, the work suggests advancements in longevity and health events useful for life insurance schemes. However, the paper's lack of mathematical derivation limits its actuarial depth. It seems to lack technical depth concerning smoothing methods and does not advance stochastic mortality projection. The mortality data which reflects insured lives is susceptible selection effects and probably incorrectly estimates mortality rates in comparison to the general population. Although it is useful in core actuarial valuation, it reflects restricted technical advancement in mortality estimation. Heligman and Pollard (1980) developed influential parsimonious mortality functions in actuarial modelling. Their parametric formulae integrate three additive mortality components: infant mortality, accidental hump and adult mortality. The function's basic application is the potential to enumerate the entire age-specific spectrum of mortality with a structured parsimonious form. It enhances smooth mortality graduation and interpretable

parameters associated with biological processes making it a useful tool in mortality frameworks. Nonetheless, the intractable behaviour of the mortality function is a setback. The parameters may be highly correlated causing computational instability and convergence problems. It is static and does not implicitly account for time trends in mortality projection. Moreover, overfitting can occur in small samples making the mortality model not to perform well in populations with sparse mortality trends as a result of epidemics or mortality shocks.

Debon *et al.* (2005) investigates many parsimonious mortality models employing data from Valencia region. The main contribution is technical benchmarking valuating diverse mortality models which includes Heligman-Pollard's law employing goodness of fit diagnostics. The authors offer guide on model selection in applied actuarial frameworks by using different specifications. Its weaknesses are geographical specifications as its conclusions are based on Spanish mortality data and consequently this cannot not generalise global applications. The study emphasised more on in-sample fit rather than out-of-sample projection performance. The comparison is basically frequentist and does not investigate Bayesian or stochastic mortality models.

Chauhan (2002) doctoral work offer major contributions to mortality projection in the Indian context by employing both statistical time-series and demographic methods. The author highlights structural demographic changes in third world nations and focus on projection in policy planning. The major application borders on theoretical mortality modeling in the emerging economies where mortality data quality and demographic transition dynamics are different from those of advanced economies. Nevertheless, the technical estimation tools are relatively old in comparison to current stochastic mortality models. The work predates some Bayesian projection methods. Sparse data problems and quality issues are constraints to model robustness.

Edrus *et al.* (2021) compares parsimonious Heligman-Pollard's mortality function with non-parsimonious P-spline mortality smoothing in Malaysian mortality table. The authors' contribution centers on numerically evaluating deviation between structured parsimonious mortality functions and spline smoothing methods. The results emphasises that P-splines can offer robust fit in defined age range as a result of the model's flexibility. The weakness is that the discussions remain on fitting rather than long-term projection performance. The paper is further restricted to Malaysian mortality data and does not deeply investigate uncertainty quantification.

Ibrahim and Nordin (2020) associates mortality improvement with pension cost sustainability. The major contribution is investigating how longevity improvements substantially increase pension liabilities. The incorporation of mortality projection to actuarial valuation enhances substantial importance to mortality estimations. The problem is that mortality improvement assumptions rely much on deterministic projections. The paper does not comprehensively offer stochastic mortality uncertainty or parameter risk that are critical in actuarial pension risk management.

Umar and Chukwudi (2019) compares Heligman-Pollard and Lee-Carter models employing Nigerian mortality data. Its contributions are both structural parsimonious and stochastic factor-based models in a developing nation context. It explains gaps between age-specific fitting and time-dynamic projection Lee-Carter. The weaknesses are potential data reliability problems and restricted exploration of model uncertainty. The author does not deal with multi-population extensions or Bayesian refinements.

Silva, Ramos & Olvera (2020) deploys Heligman-Pollard to Mexican mortality data with inferential emphasis. The main contribution is statistical inference on Heligman-Pollard parameters while improving demographic interpretation and confidence intervals. The weakness is its total emphasis on Heligman-Pollard without wider model comparisons. It remains basically cross-sectional and does not integrate time-series forecasting. Mazzuco, Scarpa and Zanotto

(2018) suggests a mixture distribution technique to mortality estimation. The main strength is analytic innovation incorporating mortality heterogeneity using latent subpopulations. This method offers much flexibility than single-form parsimonious laws. Nevertheless, mixture mortality models are computationally intensive and may experience identifiability problems. The interpretation of mixture mortality terms may lack obvious actuarial meaning in relation to parsimonious models like Heligman-Pollard's models.

Ibrahim *et al.* (2017) extends Heligman-Pollard's models into forecasting Malaysian mortality rates. The strength lies in adopting static parsimonious model into a time-structured projection frameworks. The weakness is that Heligman-Pollard's model is not implicitly formulated for stochastic time dependent dynamics, likely inhibiting predictive robustness relative to Lee-Carter models. Gaille (2012) links actuarial mortality models and demographic analysis. The contribution lies in deep implementation, model risk and regulatory concerns. The work is not a technical innovation.

Ozeki (2005) employs mortality models to Japanese data, a high-longevity population. The contribution is a theoretical validation in a super-aging society. The paper is limited in technical novelty and focus much on theoretical rather than application. Chanco (2016) evaluates Whittaker-Henderson

Mortality Model.

Let $F_{T_x}(\zeta)$ be the probability distribution function of the random life time T_x for an insured aged x , then

$$F_{T_x}(\zeta) = \mathbf{P}(T_x \leq \zeta) = {}_\zeta q_x \quad (1)$$

where $F_{T_x}(\zeta)$ specifies the probability that a life aged x dies before reaching age $x + \zeta$. The complimentary probability of survival function $S_{T_x}(\zeta)$ is

$${}_ \zeta p_x = S_{T_x}(\zeta) = 1 - F_{T_x}(\zeta) = \mathbf{P}(T_x > \zeta) \quad (2)$$

The probability density function $f_{T_x}(\zeta)$ of the random life time T_x is

$$f_{T_x}(\zeta) = \frac{d}{d\zeta} F_{T_x}(\zeta) = \frac{d}{d\zeta} (1 - S_{T_x}(\zeta)) = \frac{d}{d\zeta} (-S_{T_x}(\zeta)) \quad (3)$$

The probability that an insured aged x dies within ζ further years defines the probability that a life aged $x = 0$ dies within $x + \zeta$ years provided that they survive x years

$${}_ \zeta q_x = \mathbf{P}(T_x \leq \zeta) = \mathbf{P}(T_0 \leq x + \zeta | T_0 \leq x) \quad (4)$$

graduation for mortality estimation. The contribution reinforces classical actuarial smoothing methods. The weakness is that Whittaker-Henderson is deterministic and is not implicitly useful in stochastic forecasting. Atkinson and McGarry (2016) offers useful guide in constructing mortality experience studies. Its contribution is technically rigorous in exposure computations and credibility analysis. However, the emphasis is more on actuarial operations rather than mortality modelling.

Putri, Siagian & Soleh (2023) deploys Makeham's model to Takaful insurance contribution estimations. Its contribution covers incorporating classical mortality laws into Sharia actuarial practice. The paper relies too much on simple mortality laws which do not account for complicated age patterns.

Njenga (2020) suggested Bayesian VAR approaches in mortality estimations. Although the paper incorporates uncertainty and multivariate dependent dynamics, allowing projection robustness, its limitations include estimation problems and model specification sensitivity.

Rusli (2024) compares Whittaker-Henderson and Heligman-Pollard in mortality smoothing. The contribution focuses on evaluating parametric and graduation methods using Indonesian mortality data. The paper is restricted to mortality graduation rather than stochastic projection.

$$\mathbf{P}(T_0 \leq x + \zeta | T_0 \leq x) = \frac{\mathbf{P}(x < T_0 \leq x + \zeta)}{\mathbf{P}(T_0 > x)} \quad (5)$$

This expression is equal to

$${}_{\zeta}q_x = F_{T_x}(x) = \frac{F_{T_0}(x + \zeta) - F_{T_0}(x)}{S_{T_0}(x)} \quad (6)$$

$${}_{\zeta}q_x = F_{T_x}(x) = \frac{1 - S_{T_0}(x + \zeta) - (1 - S_{T_0}(x))}{S_{T_0}(x)} \quad (7)$$

$${}_{\zeta}q_x = F_{T_x}(x) = \frac{1 - S_{T_0}(x + \zeta) - 1 + S_{T_0}(x)}{S_{T_0}(x)} = \frac{S_{T_0}(x) - S_{T_0}(x + \zeta)}{S_{T_0}(x)} \quad (8)$$

But recall that

$${}_{\zeta}p_x = S_{T_x}(\zeta) = \frac{S_{T_0}(x + \zeta)}{S_{T_0}(x)} \quad (9)$$

The mortality rate intensity μ_x for a life insured at age x and it is denoted as

$$\mu_x = \lim_{\delta x \rightarrow 0} \left(\frac{1}{\delta x} \mathbf{P}(T_0 \leq x + \delta x | T_0 > x) \right) \quad (10)$$

$$\mu_x = \lim_{\delta x \rightarrow 0} \left(\frac{1}{\delta x} \mathbf{P}(T_x \leq \delta x) \right) = \lim_{\delta x \rightarrow 0} \left(\frac{1}{\delta x} \times ({}_{\delta x}q_x) \right) \quad (11)$$

$$\mu_x = \lim_{\delta x \rightarrow 0} \left(\frac{1}{\delta x} (1 - S_{T_x}(\delta x)) \right) \quad (12)$$

$$\mu_x = \lim_{\delta x \rightarrow 0} \left(\frac{1}{\delta x} \left(1 - \frac{S_{T_0}(x + \zeta)}{S_{T_0}(x)} \right) \right) \quad (13)$$

$$\mu_x = \lim_{\delta x \rightarrow 0} \left(\frac{1}{\delta x} \left(\frac{S_{T_0}(x) - S_{T_0}(x + \zeta)}{S_{T_0}(x)} \right) \right) \quad (14)$$

$$\mu_x = \frac{1}{S_{T_0}(x)} \lim_{\delta x \rightarrow 0} \left(\left(\frac{S_{T_0}(x) - S_{T_0}(x + \zeta)}{\delta x} \right) \right) \quad (15)$$

$$\mu_x = \frac{1}{S_{T_0}(x)} \left(-\frac{d}{dx} S_{T_0}(x) \right) \quad (16)$$

$$-\mu_x = \frac{1}{S_{T_0}(x)} \left(\frac{d}{dx} S_{T_0}(x) \right) \quad (17)$$

$$\mu_x = \frac{f_{T_0}(x)}{S_{T_0}(x)} \quad (18)$$

Observe that

$$\frac{d}{d\zeta}(x + \zeta) = d\zeta \quad (19)$$

$$\mu_{x+\zeta} = \frac{-1}{S_{T_0}(x+\zeta)} \left(\frac{d}{d(x+\zeta)} S_{T_0}(x+\zeta) \right) \quad (20)$$

$$\mu_{x+\zeta} = \frac{-1}{S_{T_0}(x+\zeta)} \left(\frac{d}{d\zeta} S_{T_0}(x+\zeta) \right) \quad (21)$$

But recall that

$$S_{T_0}(x+\zeta) = S_{T_0}(\zeta) S_{T_x}(\zeta) \quad (22)$$

$$\mu_{x+\zeta} = \frac{-1}{S_{T_0}(x) S_{T_x}(\zeta)} \left(\frac{d}{d\zeta} S_{T_x}(\zeta) S_{T_0}(x) \right) \quad (23)$$

$$\mu_{x+\zeta} = \frac{-1}{S_{T_x}(\zeta)} \left(\frac{d}{d\zeta} S_{T_x}(\zeta) \right) \quad (24)$$

$$\mu_{x+\zeta} = \frac{f_{T_x}(\zeta)}{S_{T_x}(\zeta)} \quad (25)$$

$$\mu_{x+\zeta} = - \left(\frac{d}{d\zeta} \log_e S_{T_x}(\zeta) \right) \quad (26)$$

$$\int_0^u \mu_{x+\zeta} d\zeta = \log_e S_{T_x}(0) - \log_e S_{T_x}(u) \quad (27)$$

$$\int_0^u \mu_{x+\zeta} d\zeta = \log_e 1 - \log_e S_{T_x}(u) \quad (28)$$

$$\int_0^u \mu_{x+\zeta} d\zeta = -\log_e S_{T_x}(u) \quad (29)$$

$$\exp \left(- \int_0^u \mu_{x+\zeta} d\zeta \right) = S_{T_x}(u) \quad (30)$$

$${}_u p_{x+0} = \exp \left(- \int_0^u \mu_{x+\zeta} d\zeta \right) \quad (31)$$

The Incomplete Gamma Survival Function

According to Khan, Rakibul, Usman (2024), the incomplete Gamma function is defined in (31a-b) as

$$B_{\rho}^{(\alpha, \beta; m)}(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} {}_1F_1 \left(\alpha : \beta; \frac{-\rho}{t^m (1-t)^m} \right) dt \quad (31a)$$

$$B_{\rho}^{(\alpha, \beta)}(u, v) = B_{\rho}^{(\alpha, \beta; 1)}(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} {}_1F_1 \left(\alpha : \beta; \frac{-\rho}{t(1-t)} \right) dt; \quad m=1 \quad (31b)$$

We now show that the hypergeometric function ${}_1F_1(.,.,.)$ is equal to (31c)

$${}_1F_1 \left(\alpha : \beta; \frac{-\rho}{t(1-t)} \right) = \exp \left(\frac{-\rho}{t(1-t)} \right) \text{ if } \alpha = \beta \quad (31c)$$

$${}_1F_1(\alpha : \beta; q) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\beta)_m} \frac{q^m}{m!} \quad (31d)$$

$$(\alpha)_m = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\dots(\alpha+m-1) \quad (31e)$$

$$(\alpha)_0 = 1 \quad (31f)$$

$${}_1F_1(\alpha : \alpha; q) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\alpha)_m} \frac{q^m}{m!} \quad (31g)$$

$${}_1F_1(\alpha : \alpha; q) = \sum_{m=0}^{\infty} \frac{q^m}{m!} \quad (31h)$$

$${}_1F_1(\alpha : \alpha; q) = \sum_{m=0}^{\infty} \frac{q^m}{m!} \quad (31i)$$

$${}_1F_1(\alpha : \alpha; q) = \exp(q) \quad (31j)$$

$$q = \frac{-\rho}{t(1-t)} \quad (31k)$$

$${}_1F_1(\alpha : \alpha; q) = \exp\left(\frac{-\rho}{t(1-t)}\right) \quad (31l)$$

$$B_0^{(\alpha, \alpha; 1)}(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} \exp\left(\frac{0}{t(1-t)}\right) dt \quad (31m)$$

$$B_0^{(\alpha, \alpha; 1)}(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} \times 1 dt = B(u, v) \quad (31n)$$

$$\gamma(A, x) = \int_0^x t^{A-1} e^{-t} dt \quad (32)$$

$$\Gamma(A, x) = \int_x^{\infty} t^{A-1} e^{-t} dt \quad (33)$$

$$\Gamma(A) = \int_0^{\infty} t^{A-1} e^{-t} dt \quad (34)$$

$$\gamma(A, x) = x^A e^{-x} \sum_{m=0}^{\infty} \int_0^1 (1-y)^{A-1} \frac{(x \times y)^m}{m!} dy \quad (35)$$

$$\gamma(A, x) = x^A e^{-x} \sum_{m=0}^{\infty} \frac{x^m}{m!} \int_0^1 (1-y)^{A-1} y^m dy \quad (36)$$

$$\gamma(A, x) = x^A e^{-x} \sum_{m=0}^{\infty} \frac{x^m}{m!} B(A, m+1) \quad (37)$$

$$B(A, B) = \int_0^1 t^{A-1} (1-t)^{B-1} dt \quad (38)$$

$$\gamma(A, x) = x^A e^{-x} \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(m+1)} \frac{\Gamma(A) \times \Gamma(m+1)}{\Gamma(A+m+1)} \quad (39)$$

$$\gamma(A, x) = e^{-x} \sum_{m=0}^{\infty} x^{m+A} \frac{\Gamma(A)}{\Gamma(A+m+1)} \quad (40)$$

$$\gamma(A, x) = e^{-x} \sum_{m=0}^{\infty} x^{m+A} \frac{\Gamma(A)}{\Gamma(A+m+1)} \quad (41)$$

$$\gamma(A, x) = e^{-x} \sum_{m=0}^{\infty} \frac{x^{m+A}}{(A)_{m+1}} \quad (42)$$

$$(A)_m = \frac{\Gamma(m+A)}{\Gamma(A)} \quad (43)$$

$$\gamma(A, x) + \Gamma(A, x) = \Gamma(A) \quad (44)$$

$$e^{-x} \sum_{m=0}^{\infty} \frac{x^{m+A}}{(A)_{m+1}} + \Gamma(A, x) = \Gamma(A) \quad (45)$$

$$\Gamma(A, x) = \Gamma(A) - e^{-x} \sum_{m=0}^{\infty} \frac{x^{m+A}}{(A)_{m+1}} \quad (46)$$

Materials and Methods

The Survival Probability and the Heligman-Pollard's Third Law Theorem

Given that the force of mortality is

$$\mu_x = A_1^{(x+A_2)^{A_3}} + B_1 \exp\left(-B_2 (\log_e x - \log_e B_3)^2\right) + \frac{C_1 C_2^x}{1 + C_3 C_1 C_2^x} \quad (47)$$

then,

$${}_t p_x = \exp \left\{ \left[\frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \right. \right. \quad (48)$$

$$\left. \left[\frac{B_1 (x+t) e^{\left[-B_2 [\log_e (x+t)]^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e (x+t) \right]}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e (x+t)\}} \right] \right. \quad (48)$$

$$\left. - \left[\frac{B_1 x e^{\left(-B_2 (\log_e x)^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right)}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e x\}} \right] \right. \quad (48)$$

$$\left. + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) \right] \right] \quad (48)$$

Proof

Expanding the second law in Heligman and Pollard (1980) at arbitrary time ζ yields

$$\mu_{x+\zeta} = A_1^{(x+\zeta+A_2)^{A_3}} + B_1 \exp \left[-B_2 \left(\log_e (x+\zeta) - \log_e B_3 \right)^2 \right] + \frac{C_1 C_2^{x+\zeta}}{1 + C_3 C_1 C_2^{x+\zeta}} \quad (49)$$

$$\mu_{x+\zeta} = A_1^{(x+\zeta+A_2)^{A_3}} + B_1 e^{-B_2 \left[\left(\log_e (x+\zeta) - \log_e B_3 \right)^2 \right]} + C_1 C_2^{x+\zeta} \quad (50)$$

$$\int_0^t \mu(x+s) ds = \int_0^t \left\{ A_1^{(x+s+A_2)^{A_3}} + B_1 \exp \left(-B_2 \log_e \left(\frac{x+s}{B_3} \right)^2 \right) + \frac{C_1 C_2^{x+s}}{1 + C_3 C_1 C_2^{x+s}} \right\} ds \quad (51)$$

$$\mu_{x+\zeta} = A_1^{(x+\zeta+A_2)^{A_3}} + B_1 e^{-B_2 \left[\left(\log_e (x+\zeta) - \log_e B_3 \right)^2 \right]} + \frac{C_1 C_2^{x+\zeta}}{1 + C_3 C_1 C_2^{x+\zeta}} \quad (52)$$

The auxiliary component mortality intensities can be expressed as

$$\mu_1(x+\zeta) = A_1^{(x+\zeta+A_2)^{A_3}} \quad (53)$$

$$\mu_2(x+\zeta) = B_1 e^{-B_2 \left[\left(\log_e (x+\zeta) - \log_e B_3 \right)^2 \right]} \quad (54)$$

$$\mu_3(x+\zeta) = \frac{C_1 C_2^{x+\zeta}}{1 + C_3 C_1 C_2^{x+\zeta}} \quad (55)$$

Following methodologies in Strab (2018) and using equation (53), define the transformation

$$\psi = (x+s+A_2)^{A_3} \quad (56)$$

$$\psi^{\frac{1}{A_3}} = (x+s+A_2) \quad (57)$$

Then, this implies

$$\psi^{\frac{1}{A_3}} = (x+s+A_2)^{-1} \quad (58)$$

Differentiating ψ using (56) yields

$$\frac{d\psi}{ds} = A_3 (x+s+A_2)^{A_3-1} = A_3 (x+s+A_2)^{A_3} (x+s+A_2)^{-1} \quad (59)$$

$$\frac{d\psi}{ds} = A_3 \frac{\psi}{(x+s+A_2)} = A_3 \psi \frac{1}{\psi^{\frac{1}{A_3}}} = A_3 \psi^{1-\frac{1}{A_3}} \Rightarrow d\psi = A_3 \psi^{1-\frac{1}{A_3}} ds \quad (60)$$

$$\frac{d\psi}{A_3 \psi^{1-\frac{1}{A_3}}} = ds \quad (61)$$

$$\mu_1(x+\zeta) = A_1^\psi \quad (62)$$

Integrate the first mortality component with respect to arbitrary time s to obtain

$$\Lambda_x = \int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \int_{(x+A_2)^{A_3}}^{(x+t+A_2)^{A_3}} \left(\frac{A_1^\psi}{A_3 \psi^{1-\frac{1}{A_3}}} \right) d\psi \quad (63)$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \int_{(x+A_2)^{A_3}}^{(x+t+A_2)^{A_3}} \left(\frac{e^{\psi \log_e A_1}}{A_3 \psi^{1-\frac{1}{A_3}}} \right) d\psi \quad (64)$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \int_{(x+A_2)^{A_3}}^{(x+t+A_2)^{A_3}} \left(\psi^{-1+\frac{1}{A_3}} e^{\psi \log_e A_1} \right) d\psi \quad (65)$$

Effect a second change of variable by letting,

$$\xi = \log_e \left(\frac{1}{A_1} \right)^\psi = \psi \log_e \left(\frac{1}{A_1} \right) = \psi \log_e (A_1)^{-1} = -\psi \log_e A_1 \quad (66)$$

$$\xi = -\psi \log_e A_1 \Rightarrow \xi = \log_e A_1^{-\psi} \Rightarrow e^\xi = A_1^{-\psi} \Rightarrow e^{-\xi} = A_1^\psi \quad (67)$$

$$\xi = -\psi \log_e A_1 \Rightarrow \frac{-\xi}{\log_e A_1} = \psi \quad (68)$$

$$\frac{d\xi}{d\psi} = -\log_e A_1 \Rightarrow \frac{d\xi}{-\log_e A_1} = d\psi \quad (69)$$

From equation (67), when

$$\psi = (x + A_2)^{A_3} \quad (70)$$

$$\xi = -(x + A_2)^{A_3} (\log_e A_1) \quad (71)$$

However, when

$$\psi = (x + t + A_2)^{A_3} \quad (72)$$

$$\xi = -(x + t + A_2)^{A_3} (\log_e A_1) \quad (73)$$

Therefore, equation (65) transforms to (53)

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \int_{(x+A_2)^{A_3}}^{(x+t+A_2)^{A_3}} \left(\frac{A_1^\psi}{A_3 \psi^{1-\frac{1}{A_3}}} \right) d\psi \quad (74)$$

$$= \frac{-1}{A_3} \int_{-(x+A_2)^{A_3} \log_e A_1}^{-(x+t+A_2)^{A_3} \log_e A_1} \left(\left(\frac{-\xi}{\log_e A_1} \right)^{-1+\frac{1}{A_3}} e^{-\xi} \right) \frac{d\xi}{\log_e A_1}$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{-1}{A_3} \times \frac{1}{\log_e A_1} \int_{-(x+A_2)^{A_3} \log_e A_1}^{-(x+t+A_2)^{A_3} \log_e A_1} \left(\left(\frac{-1}{\log_e A_1} \right)^{-1+\frac{1}{A_3}} (\xi)^{-1+\frac{1}{A_3}} e^{-\xi} \right) d\xi \quad (75)$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{-1}{A_3} \times \left(\frac{1}{-\log_e A_1} \right)^{-1} \times \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \times \frac{1}{\log_e A_1} \int_{-(x+A_2)^{A_3} \log_e A_1}^{-(x+t+A_2)^{A_3} \log_e A_1} \left((\xi)^{-1+\frac{1}{A_3}} e^{-\xi} \right) d\xi \quad (76)$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{-1}{A_3} \times (-\log_e A_1) \times \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \times \frac{1}{\log_e A_1} \int_{-(x+A_2)^{A_3} \log_e A_1}^{-(x+t+A_2)^{A_3} \log_e A_1} \left((\xi)^{-1+\frac{1}{A_3}} e^{-\xi} \right) d\xi \quad (77)$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \times \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \times \int_{-(x+A_2)^{A_3} \log_e A_1}^{-(x+t+A_2)^{A_3} \log_e A_1} \left((\xi)^{\frac{1}{A_3}-1} e^{-\xi} \right) d\xi \quad (78)$$

$\gamma(z, x) = \int_0^x t^{z-1} e^{-t} dt$ is the incomplete lower Gamma integral

$\Gamma(z, x) = \int_x^\infty t^{z-1} e^{-t} dt$ is the incomplete upper Gamma integral

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma integral

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \times \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \times \int_{-(x+A_2)^{A_3} \log_e A_1}^{-(x+t+A_2)^{A_3} \log_e A_1} \left((\xi)^{\frac{1}{A_3}-1} e^{-\xi} \right) d\xi \quad (79)$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \times \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \times \left\{ \begin{aligned} &\int_{(-1)(x+A_2)^{A_3} \log_e A_1}^\infty \left(e^{-\xi} (\xi)^{\frac{1}{A_3}-1} \right) d\xi \\ &-\int_{(-1)(x+t+A_2)^{A_3} \log_e A_1}^\infty \left(e^{-\xi} (\xi)^{\frac{1}{A_3}-1} \right) d\xi \end{aligned} \right\} \quad (80)$$

$$\int_0^t A_1^{(x+s+A_2)^{A_3}} ds = \frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \times \left\{ \begin{aligned} &\Gamma\left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1}\right) \\ &-\Gamma\left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1}\right) \end{aligned} \right\} \quad (81)$$

Let $z = \frac{1}{A_3} > 0$ and

$$y = (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \quad (82)$$

But in mortality, the gamma function can be defined as

$$G(z, y) = \frac{1}{\Gamma(z)} \int_0^y t^{z-1} e^{-t} dt \quad (83)$$

Consequently,

$$\Gamma(z, y) = \Gamma(z) - \Gamma(z) \times G(z, y) \quad (84)$$

Integrating the second term teenage mortality component with respect to arbitrary time s ,

$$\mu_2(x+s) = B_1 e^{-B_2 [(\log_e(x+s) - \log_e B_3)^2]} \quad (85)$$

Expanding the exponent in equation (68) yields

$$\mu_2(x+s) = B_1 e^{[-B_2 [\log_e(x+s)]^2 - B_2 [\log_e B_3]^2 + 2B_2 (\log_e B_3) \log_e(x+s)]} \quad (86)$$

The auxiliary hazard function in equation (86),

$$\int_0^t \mu_2(x+s)ds = \left[\frac{B_1 e^{\left[-B_2 [\log_e(x+s)]^2 - B_2 [\log_e B_3]^2 + 2B_2 (\log_e B_3) \log_e(x+s) \right]}}{-2B_2 \left\{ \log_e(x+s) \right\} \times \frac{1}{(x+s)} + 2B_2 (\log_e B_3) \times \frac{1}{(x+s)}} \right]_0^t \quad (87)$$

$$\int_0^t \mu_2(x+s)ds = \left[\frac{B_1 e^{\left[-B_2 [\log_e(x+s)]^2 - B_2 [\log_e B_3]^2 + 2B_2 (\log_e B_3) \log_e(x+s) \right]}}{2B_2 (\log_e B_3) - 2B_2 \left\{ \log_e(x+s) \right\}} \times (x+s) \right]_0^t \quad (88)$$

$$\int_0^t \mu_2(x+s)ds = \left[\frac{B_1 (x+t) e^{\left[-B_2 [\log_e(x+t)]^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e(x+t) \right]}}{2B_2 (\log_e B_3) - 2B_2 \left\{ \log_e(x+t) \right\}} \right] - \left[\frac{B_1 x e^{\left(-B_2 (\log_e x)^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right)}}{2B_2 (\log_e B_3) - 2B_2 \left\{ \log_e x \right\}} \right] \quad (89)$$

The third component is

$$\mu_5(x+\zeta) = \frac{C_1 C_2^{x+\zeta}}{1 + C_3 C_1 C_2^{x+\zeta}} \quad (90)$$

Letting,

$$U = C_3 C_1 C_2^{x+\zeta} = C_3 C_1 C_2^x C_2^\zeta \Rightarrow \frac{dU}{d\zeta} = C_3 C_1 (\log_e C_2) C_2^{x+\zeta} \quad (91)$$

$$\Rightarrow dU = (\log_e C_2) C_3 C_1 C_2^{x+\zeta} d\zeta$$

$$\Rightarrow \frac{dU}{C_3 \log_e C_2} = C_1 C_2^{x+\zeta} d\zeta \quad (92)$$

Integrate the third component with respect to arbitrary time ζ in the range $[0, t]$ to obtain

$$\int_0^t \frac{C_1 C_2^{x+\zeta} d\zeta}{1 + C_3 C_1 C_2^{x+\zeta}} = \frac{1}{C_3 \log_e C_2} \int_{C_3 C_1 C_2^x}^{C_3 C_1 C_2^{x+t}} \frac{dU}{1+U} = \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) \right] - \left[-\log_e (1 + C_3 C_1 C_2^x) \right] \quad (93)$$

$$\begin{aligned}
\Rightarrow \int_0^t \mu_{x+\zeta} d\zeta &= \frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right. \\
&\quad \left. - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} \\
&+ \left[\frac{B_1(x+t) e^{\left[-B_2 [\log_e(x+t)]^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e(x+t) \right]}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e(x+t)\}} \right] \\
&- \left[\frac{B_1 x e^{\left(-B_2 (\log_e x)^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right)}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e x\}} \right] + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) \right. \\
&\quad \left. - \log_e (1 + C_3 C_1 C_2^x) \right]
\end{aligned} \tag{94}$$

$$\begin{aligned}
\text{But } {}_t p_{x+0} &= \exp \left(- \int_0^t \mu_{x+u} du \right) \\
{}_t p_x &= \exp \left\{ \begin{aligned} &\frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right. \\ &\quad \left. - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \\ &\left[\frac{B_1(x+t) e^{\left[-B_2 [\log_e(x+t)]^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e(x+t) \right]}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e(x+t)\}} \right] \\ &- \left[\frac{B_1 x e^{\left(-B_2 (\log_e x)^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right)}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e x\}} \right] \\ &+ \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) \right. \\ &\quad \left. - \log_e (1 + C_3 C_1 C_2^x) \right] \end{aligned} \right\}
\end{aligned} \tag{95}$$

$$\begin{aligned}
\Rightarrow \lim_{t \rightarrow 0} ({}_t p_x) &= \exp \left\{ \begin{aligned} &\frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right. \\ &\quad \left. - \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \\ &\left[\frac{B_1 x e^{\left[-B_2 [\log_e x]^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right]}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e(x)\}} \right] \\ &- \left[\frac{B_1 x e^{\left(-B_2 (\log_e x)^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right)}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e x\}} \right] \\ &+ \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^x) \right. \\ &\quad \left. - \log_e (1 + C_3 C_1 C_2^x) \right] \end{aligned} \right\} = 1
\end{aligned} \tag{96}$$

This is valid since all brackets cancel out and $e^0 = 1$

This completes the proof.

Next, transit from survival probability to survival function to obtain equation (97)

$${}_t p_0 = \exp - \left\{ \left[\frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (A_2)^{A_3} \log_e \frac{1}{A_1} \right) - \Gamma \left(\frac{1}{A_3}, (t + A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \right. \right. \quad (97)$$

$$\left. \left[\frac{B_1 \times t \times e^{\left[-B_2 [\log_e t]^2 - B_2 (\log_e B_3)^2 + 2 B_2 (\log_e B_3) \log_e t \right]}}{2 B_2 (\log_e B_3) - 2 B_2 \{ \log_e t \}} \right] \right. \\ \left. + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^t) \right] - \log_e (1 + C_3 C_1) \right] \right\}$$

Hence, the survival function becomes,

$$l_x = l_0 \exp - \left\{ \left[\frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (A_2)^{A_3} \log_e \frac{1}{A_1} \right) - \Gamma \left(\frac{1}{A_3}, (x + A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \right. \quad (98)$$

$$\left[\frac{B_1 \times x \times e^{\left[-B_2 [\log_e x]^2 - B_2 (\log_e B_3)^2 + 2 B_2 (\log_e B_3) \log_e x \right]}}{2 B_2 (\log_e B_3) - 2 B_2 \{ \log_e x \}} \right] \right. \\ \left. + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^x) \right] - \log_e (1 + C_3 C_1) \right] \right\}$$

This gives the number of lives surviving to age x .

The Whole Life Annuity under the Heligman-Pollard's Third Law

$$\gamma = \sum_{k=0}^{\infty} \left(\frac{1}{1+i} \right)^k 1_{\{T_x > k\}} \quad (99)$$

$$\mathbf{P} \left(1_{\{T_x > k\}} \right) = {}_k p_x \quad (100)$$

$$\mathbf{E}(\gamma) = \mathbf{E} \left(\sum_{k=0}^{\infty} \left(\frac{1}{1+i} \right)^k 1_{\{T_x > k\}} \right) \quad (101)$$

$$\mathbf{E}(\gamma) = \sum_{k=0}^{\infty} \mathbf{E} \left(\left(\frac{1}{1+i} \right)^k 1_{\{T_x > k\}} \right) \quad (102)$$

$$\mathbf{E}(\gamma) = \sum_{k=0}^{\infty} \left(\left(\frac{1}{1+i} \right)^k \mathbf{E}(1_{\{T_x > k\}}) \right) \quad (103)$$

$$\mathbf{E}(\gamma) = \sum_{k=0}^{\infty} \left(\left(\frac{1}{1+i} \right)^k \mathbf{P}(1_{\{T_x > k\}}) \right) \quad (104)$$

$$\mathbf{E}(\gamma) = \sum_{k=0}^{\infty} (v_{\zeta}^k p_x) \quad (105)$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} ({}_k|_1 q_x) \quad (106)$$

$$\ddot{a}_x = \sum_{t=0}^{\Omega-x-1} v^t \left\{ \exp - \left\{ \begin{aligned} & \frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right. \right. \\ & \left. \left. - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \right. \\ & \left[\frac{B_1(x+t) e^{\left[-B_2 [\log_e(x+t)]^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e(x+t) \right]}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e(x+t)\}} \right] \\ & - \left[\frac{B_1 x e^{\left(-B_2 (\log_e x)^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right)}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e x\}} \right] \\ & + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) \right. \\ & \left. \left. - \log_e (1 + C_3 C_1 C_2^x) \right] \right\} \right\} \quad (107)$$

The continuously incepted whole life annuity is given by equation (108)

$$\bar{\mathbf{a}}_x = \sum_{t=0}^{\Omega-x-1} v^t \left\{ \exp - \left\{ \begin{aligned} & \frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right. \right. \\ & \left. \left. - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \right. \\ & \left[\frac{B_1(x+t) e^{\left[-B_2 [\log_e(x+t)]^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e(x+t) \right]}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e(x+t)\}} \right] \\ & - \left[\frac{B_1 x e^{\left(-B_2 (\log_e x)^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right)}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e x\}} \right] \\ & + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) \right. \\ & \left. \left. - \log_e (1 + C_3 C_1 C_2^x) \right] \right\} \right\} \\ & - \frac{1}{2} - \frac{1}{12} \left[\log_e (1+i) + A_1^{(x+A_2)^{A_3}} + B_1 \exp \left(-B_2 (\log_e x - \log_e B_3)^2 \right) + \frac{C_1 C_2^x}{1 + C_3 C_1 C_2^x} \right] \quad (108)$$

The life whole annuity payable payable m times is defined in (109)

$$\ddot{a}_x^{(K)} = \sum_{t=0}^{\Omega-x-1} v^t \left\{ \exp - \left\{ \begin{aligned} & \frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right. \\ & \left. - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \\ & \left[\frac{B_1 (x+t) e^{\left[-B_2 [\log_e (x+t)]^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e (x+t) \right]}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e (x+t)\}} \right] \\ & - \left[\frac{B_1 x e^{\left(-B_2 (\log_e x)^2 - B_2 (\log_e B_3)^2 + 2B_2 (\log_e B_3) \log_e x \right)}}{2B_2 (\log_e B_3) - 2B_2 \{\log_e x\}} \right] \\ & + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) \right] \\ & - \log_e (1 + C_3 C_1 C_2^x) \end{aligned} \right\} \right\} \\ + \frac{K-1}{2K} - \frac{K^2-1}{12K^2} \left[\begin{aligned} & \log_e (1+i) + A_1^{(x+A_2)^{A_3}} + B_1 \exp \left(-B_2 (\log_e x - \log_e B_3)^2 \right) \\ & + \frac{C_1 C_2^x}{1 + C_3 C_1 C_2^x} \end{aligned} \right] \quad (109)$$

The Heligman-Pollard's Third Law Whole Life Insurance Function

The life insurance function defined by equation (112) or yielding equation (114)

Let $Z = v^{K_x+1}$ and $K_x^{(m)} = \frac{\lfloor mT_x \rfloor}{m}$ so that $K_x = K_x^{(1)} = \frac{\lfloor 1 \times T_x \rfloor}{1} = T_x$

$$\mathbf{E}(Z) = \mathbf{E}(v^{K_x+1}) \quad (110)$$

$$\mathbf{E}(Z) = \sum_{k=0}^{\infty} \left(\frac{1}{1+i} \right)^{k+1} \mathbf{P}(K_x = k) \quad (111)$$

$$\mathbf{E}(Z) = \sum_{k=0}^{\infty} \left(\frac{1}{1+i} \right)^{k+1} ({}_k|_1 q_x) \quad (112)$$

$$\mathbf{P}(Z \leq z) = \mathbf{P} \left(K_x \geq \frac{-(\delta + \log_e z)}{\delta} \right) \quad (113)$$

$$A_x = 1 - d \sum_{t=0}^{\Omega-x-1} v^t \left\{ \exp - \left[\frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \frac{B_1(x+t) e^{\left[-B_2[\log_e(x+t)]^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e(x+t) \right]}}{2B_2(\log_e B_3) - 2B_2\{\log_e(x+t)\}} - \frac{B_1 x e^{\left[-B_2(\log_e x)^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e x \right]}}{2B_2(\log_e B_3) - 2B_2\{\log_e x\}} + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) - \log_e (1 + C_3 C_1 C_2^x) \right] \right] \right\} \quad (114)$$

The net premium policy value on a policy in force within period $\zeta \geq 0$ years after it has been incepted represents the expected value at the time of the net future loss random variable on a defined basis that does not incorporate any allowance for expenses. The premiums applied in the computation are the net premiums estimated on the policy value basis using the equivalence principle and not the actual premium paid.

$$\ddot{a}_x(HP3) = \sum_{t=0}^{119-x} v^t \left\{ \exp - \left[\frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \frac{B_1(x+t) e^{\left[-B_2[\log_e(x+t)]^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e(x+t) \right]}}{2B_2(\log_e B_3) - 2B_2\{\log_e(x+t)\}} - \frac{B_1 x e^{\left[-B_2(\log_e x)^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3)\log_e x \right]}}{2B_2(\log_e B_3) - 2B_2\{\log_e x\}} + \frac{1}{C_3 \log_e C_2} \left[\log_e (1 + C_3 C_1 C_2^{x+t}) - \log_e (1 + C_3 C_1 C_2^x) \right] \right] \right\} \quad (115)$$

From the equivalence principle, the net premium is given by

$$P_x(A_x) = \left[1 - d \sum_{t=0}^{\Omega-x-1} v^t \left\{ \exp - \left\{ \frac{1}{A_3} \left(\frac{1}{-\log_e A_1} \right)^{\frac{1}{A_3}} \left\{ \Gamma \left(\frac{1}{A_3}, (x+A_2)^{A_3} \log_e \frac{1}{A_1} \right) - \Gamma \left(\frac{1}{A_3}, (x+t+A_2)^{A_3} \log_e \frac{1}{A_1} \right) \right\} + \right. \right. \right. \\
 \left. \left[\frac{B_1(x+t) e^{\left[-B_2[\log_e(x+t)]^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3) \log_e(x+t) \right]}}{2B_2(\log_e B_3) - 2B_2\{\log_e(x+t)\}} \right] \right. \\
 \left. - \left[\frac{B_1 x e^{\left(-B_2(\log_e x)^2 - B_2(\log_e B_3)^2 + 2B_2(\log_e B_3) \log_e x \right)}}{2B_2(\log_e B_3) - 2B_2\{\log_e x\}} \right] \right. \\
 \left. + \frac{1}{C_3 \log_e C_2} \left[\log_e(1 + C_3 C_1 C_2^{x+t}) \right] - \log_e(1 + C_3 C_1 C_2^x) \right] \right\} \right] \quad (116)$$

The parameters can be estimated when the sum of squares $\ell(x) = \sum_{x=0}^{\Omega} \left(\frac{\bar{q}_x}{q_x} - 1 \right)^2$ is minimised,

where $\bar{q}_x$ is the death probability from Heligman-Pollard function and q_x is the observed death probability. In executing this method, life annuity, life insurance functions and their trajectories were run on MATLAB. The parameters are obtained through the method of nonlinear optimisation using the German DAV, 2008 survivorship data

$$\begin{aligned}
 & HP_3(A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3 : male) \\
 & = \left(0.00260384, 0.085845812, 0.2578747818, 0.000703689, 6.996184702, \right. \\
 & \quad \left. 23.335423773, 0.0000520187, 1.089058747, 0.0000041 \right)
 \end{aligned}$$

$$HP_3(A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3 : \text{female})$$

$$= \begin{pmatrix} 0.001310637, 0.094226508, 0.225498727, 0.000213578, 5.989095580, \\ 17.85787705, 0.000028348, 1.087141761, -1.4674871 \end{pmatrix}$$

Presentation of Results

Table 1: Benefit Table

x	$\ddot{a}_x(M)$	$A_x(M)$	$\ddot{a}_x(F)$	$A_x(F)$	$P_x(M)$	$P_x(M)$
0	20.45609723	0.02590013	20.62958174	0.01763896	0.00126613	0.00085503
1	20.38868617	0.02911018	20.60171794	0.01896581	0.00142776	0.00092059
2	20.35137672	0.03088682	20.58053131	0.01997470	0.00151768	0.00097056
3	20.31719526	0.03251451	20.55948917	0.02097671	0.00160034	0.00102029
4	20.28301980	0.03414191	20.53784558	0.02200735	0.00168328	0.00107155
5	20.24793549	0.03581260	20.51535854	0.02307816	0.00176870	0.00112492
6	20.21156037	0.03754474	20.49190882	0.02419482	0.00185759	0.00118070
7	20.17368842	0.03934817	20.46742567	0.02536068	0.00195047	0.00123908
8	20.13418386	0.04122934	20.44188281	0.02657701	0.00204773	0.00130013
9	20.09294859	0.04319292	20.41532314	0.02784176	0.00214966	0.00136377
10	20.04992386	0.04524172	20.38788742	0.02914822	0.00225645	0.00142968
11	20.00511693	0.04737538	20.35983163	0.03048421	0.00236816	0.00149727
12	19.95864285	0.04958844	20.33154337	0.03183127	0.00248456	0.00156561
13	19.91076700	0.05186824	20.30360726	0.03316156	0.00260503	0.00163328
14	19.86193565	0.05419354	20.27704390	0.03442648	0.00272851	0.00169781
15	19.81279680	0.05653349	20.25413431	0.03551741	0.00285338	0.00175359
16	19.76423530	0.05884594	20.24194286	0.03609796	0.00297740	0.00178332
17	19.71747824	0.06107246	20.28024752	0.03427393	0.00309738	0.00169002
18	19.67438697	0.06312443	19.32602149	0.07971326	0.00320846	0.00412466
19	19.63823202	0.06484609	19.93301527	0.05080880	0.00330203	0.00254898
20	19.61591805	0.06590866	19.93082128	0.05091327	0.00335996	0.00255450
21	19.62590792	0.06543296	19.89920300	0.05241890	0.00333401	0.00263422
22	19.74159420	0.05992409	19.85797393	0.05438219	0.00303542	0.00273856
23	20.93327874	0.00317720	19.81137951	0.05660098	0.00015178	0.00285699
24	18.52363900	0.11792195	19.76081247	0.05900893	0.00636603	0.00298616
25	18.87533568	0.10117449	19.70682723	0.06157966	0.00536014	0.00312479
26	18.90029585	0.09998591	19.64966219	0.06430180	0.00529018	0.00327241
27	18.86349018	0.10173856	19.58941533	0.06717070	0.00539341	0.00342893
28	18.80248273	0.10464368	19.52611368	0.07018506	0.00556542	0.00359442
29	18.72805405	0.10818790	19.45974416	0.07334552	0.00577678	0.00376909
30	18.64436509	0.11217309	19.39026861	0.07665388	0.00601646	0.00395321
31	18.55331500	0.11650881	19.31763161	0.08011278	0.00627968	0.00414713
32	18.45586809	0.12114914	19.24176511	0.08372547	0.00656426	0.00435124
33	18.35254953	0.12606907	19.16259122	0.08749566	0.00686929	0.00456596

Actuarial Pricing Mechanisms Under Advanced Demographic Mortality Rate

34	18.24365747	0.13125441	19.08002415	0.09142742	0.00719452	0.00479179
35	18.12936336	0.13669698	18.99397167	0.09552516	0.00754009	0.00502924
36	18.00976307	0.14239223	18.90433625	0.09979351	0.00790639	0.00527887
37	17.88490462	0.14833788	18.81101599	0.10423733	0.00829403	0.00554129
38	17.75480434	0.15453313	18.71390547	0.10886164	0.00870374	0.00581715
39	17.61945677	0.16097825	18.61289646	0.11367160	0.00913639	0.00610714
40	17.47884119	0.16767423	18.50787858	0.11867245	0.00959298	0.00641200
41	17.33292633	0.17462256	18.39873994	0.12386953	0.01007462	0.00673250
42	17.18167394	0.18182505	18.28536776	0.12926820	0.01058250	0.00706949
43	17.02504159	0.18928373	18.16764885	0.13487386	0.01111796	0.00742385
44	16.86298523	0.19700070	18.04547029	0.14069189	0.01168243	0.00779652
45	16.69546122	0.20497804	17.91871989	0.14672762	0.01227747	0.00818851
46	16.52242841	0.21321769	17.78728684	0.15298634	0.01290474	0.00860088
47	16.34384984	0.22172144	17.65106224	0.15947323	0.01356605	0.00903477
48	16.15969447	0.23049074	17.50993974	0.16619335	0.01426331	0.00949137
49	15.96993869	0.23952673	17.36381615	0.17315161	0.01499860	0.00997198
50	15.77456790	0.24883010	17.21259206	0.18035276	0.01577413	0.01047795
51	15.57357783	0.25840106	17.05617258	0.18780131	0.01659227	0.01101075
52	15.36697593	0.26823924	16.89446799	0.19550152	0.01745556	0.01157193
53	15.15478266	0.27834368	16.72739448	0.20345741	0.01836672	0.01216313
54	14.93703269	0.28871273	16.55487494	0.21167262	0.01932865	0.01278612
55	14.71377607	0.29934400	16.37683972	0.22015049	0.02034447	0.01344279
56	14.48507933	0.31023432	16.19322747	0.22889393	0.02141751	0.01413516
57	14.25102648	0.32137969	16.00398595	0.23790543	0.02255134	0.01486539
58	14.01172005	0.33277524	15.80907293	0.24718700	0.02374978	0.01563577
59	13.76728184	0.34441515	15.60845703	0.25674014	0.02501693	0.01644878
60	13.51785382	0.35629268	15.40211862	0.26656578	0.02635719	0.01730709
61	13.26359877	0.36840006	15.19005076	0.27666425	0.02777527	0.01821352
62	13.00470083	0.38072853	14.97226003	0.28703524	0.02927622	0.01917114
63	12.74136601	0.39326829	14.74876747	0.29767774	0.03086547	0.02018323
64	12.47382245	0.40600845	14.51960945	0.30859003	0.03254884	0.02125333
65	12.20232061	0.41893711	14.28483854	0.31976959	0.03433258	0.02238524
66	11.92713328	0.43204127	14.04452432	0.33121313	0.03622340	0.02358308
67	11.64855539	0.44530689	13.79875417	0.34291647	0.03822851	0.02485126
68	11.36690368	0.45871887	13.54763402	0.35487457	0.04035566	0.02619458
69	11.08251613	0.47226114	13.29128903	0.36708147	0.04261317	0.02761820
70	10.79575118	0.48591661	13.02986418	0.37953028	0.04500999	0.02912772
71	10.50698681	0.49966729	12.76352480	0.39221310	0.04755572	0.03072922
72	10.21661933	0.51349432	12.49245696	0.40512110	0.05026069	0.03242926
73	9.92506194	0.52737800	12.21686785	0.41824439	0.05313599	0.03423499
74	9.63274316	0.54129794	11.93698590	0.43157210	0.05619354	0.03615419

75	9.34010493	0.55523310	11.65306086	0.44509234	0.05944613	0.03819532
76	9.04760061	0.56916188	11.36536376	0.45879220	0.06290749	0.04036758
77	8.75569268	0.58306225	11.07418659	0.47265778	0.06659236	0.04268104
78	8.46485038	0.59691189	10.77984196	0.48667419	0.07051653	0.04514669
79	8.17554703	0.61068824	10.48266249	0.50082560	0.07469693	0.04777656
80	7.88825737	0.62436870	10.18300009	0.51509523	0.07915166	0.05058384
81	7.60345468	0.63793073	9.88122498	0.52946548	0.08390012	0.05358298
82	7.32160788	0.65135201	9.57772458	0.54391788	0.08896297	0.05678989
83	7.04317851	0.66461055	9.27290220	0.55843323	0.09436230	0.06022206
84	6.76861777	0.67768487	8.96717558	0.57299164	0.10012160	0.06389879
85	6.49836357	0.69055412	8.66097516	0.58757261	0.10626585	0.06784139
86	6.23283756	0.70319821	8.35474225	0.60215513	0.11282152	0.07207345
87	5.97244244	0.71559798	8.04892705	0.61671776	0.11981664	0.07662111
88	5.71755924	0.72773527	7.74398649	0.63123874	0.12728076	0.08151341
89	5.46854485	0.73959310	7.44038194	0.64569610	0.13524495	0.08678266
90	5.22572985	0.75115572	7.13857683	0.66006777	0.14374178	0.09246490
91	4.98941645	0.76240874	6.83903418	0.67433171	0.15280519	0.09860043
92	4.75987684	0.77333920	6.54221410	0.68846600	0.16247042	0.10523440
93	4.53735180	0.78393563	6.24857124	0.70244899	0.17277383	0.11241754
94	4.32204964	0.79418811	5.95855222	0.71625942	0.18375266	0.12020696
95	4.11414553	0.80408831	5.67259320	0.72987651	0.19544479	0.12866717
96	3.91378114	0.81362947	5.39111746	0.74328012	0.20788834	0.13787125
97	3.72106466	0.82280644	5.11453311	0.75645080	0.22112124	0.14790222
98	3.53607118	0.83161566	4.84323106	0.76936995	0.23518069	0.15885469
99	3.35884342	0.84005508	4.57758309	0.78201985	0.25010248	0.17083684
100	3.18939277	0.84812415	4.31794033	0.79438379	0.26592026	0.18397285
101	3.02770065	0.85582378	4.06463204	0.80644609	0.28266460	0.19840568
102	2.87372013	0.86315618	3.81796484	0.81819215	0.30036195	0.21430060
103	2.72737785	0.87012486	3.57822249	0.82960845	0.31903348	0.23184932
104	2.58857608	0.87673447	3.34566633	0.84068256	0.33869372	0.25127508
105	2.45719509	0.88299071	3.12053671	0.85140301	0.35934905	0.27283865
106	2.33309563	0.88890021	2.90305546	0.86175926	0.38099605	0.29684561
107	2.21612168	0.89447040	2.69343012	0.87174142	0.40361971	0.32365474
108	2.10610358	0.89970935	2.49186052	0.88133998	0.42719141	0.35368753
109	2.00286173	0.90462563	2.29854890	0.89054529	0.45166654	0.38743804
110	1.90621170	0.90922801	2.11371573	0.89934687	0.47698165	0.42548147
111	1.81597244	0.91352512	1.93762497	0.90773214	0.50305010	0.46847669
112	1.73198165	0.91752468	1.77062568	0.91568449	0.52975427	0.51715306
113	1.65412861	0.92123197	1.61322400	0.92317981	0.55692887	0.57225767
114	1.58243065	0.92464616	1.46621499	0.93018024	0.58432018	0.63440917
115	1.51721985	0.92775144	1.33094137	0.93662184	0.61148122	0.70372885

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116	1.45960878	0.93049482	1.20985016	0.94238809	0.63749604	0.77892959
117	1.41261397	0.93273267	1.10785112	0.94724518	0.66028844	0.85502931
118	1.38338916	0.93412433	1.03645878	0.95064482	0.67524335	0.91720466

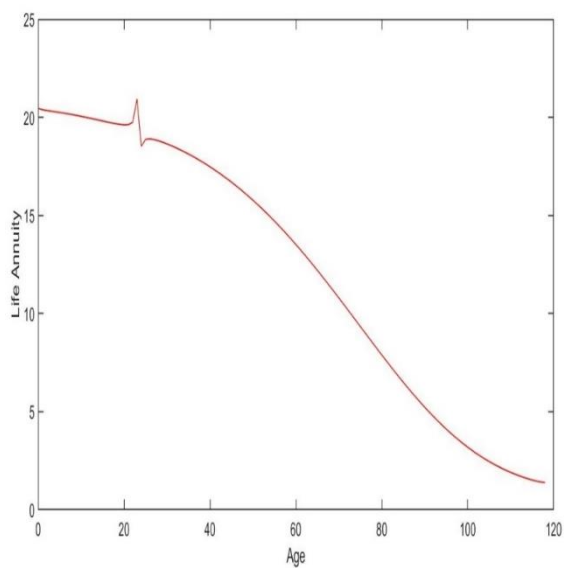


Figure 1 Male HP 3 Life Annuity

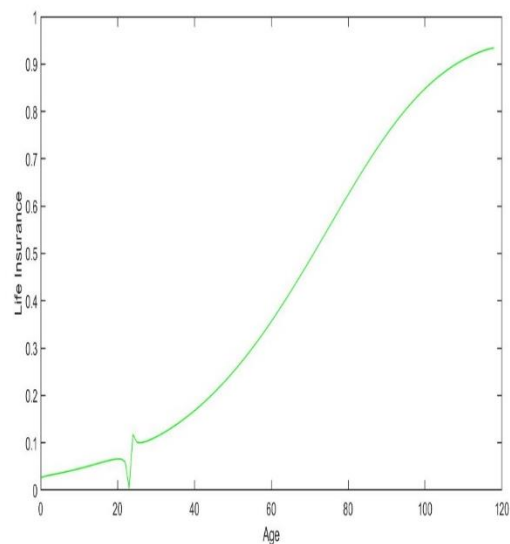


Figure 2 Male HP 3 Life Insurance

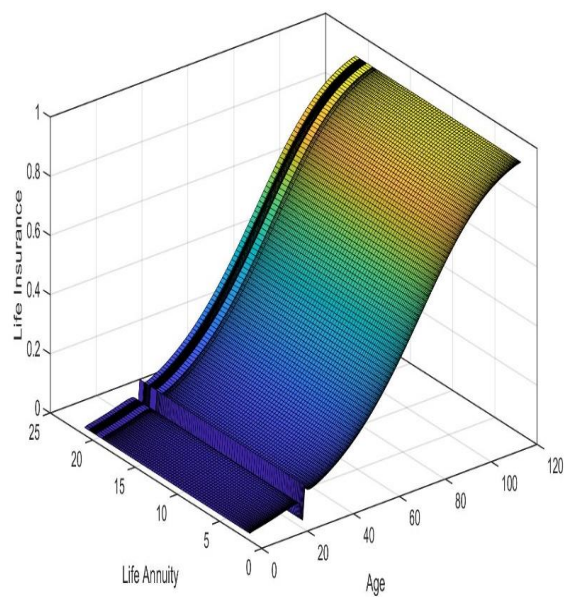


Figure 3 Male HP 3 Surface for Life Insurance and Life Annuity

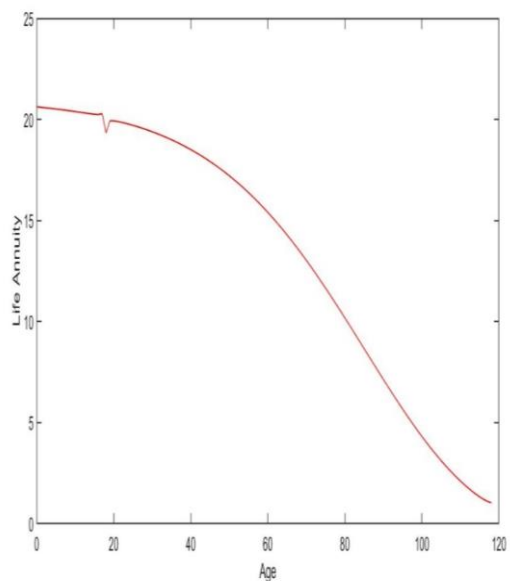


Figure 4 Female HP 3 Life Annuity

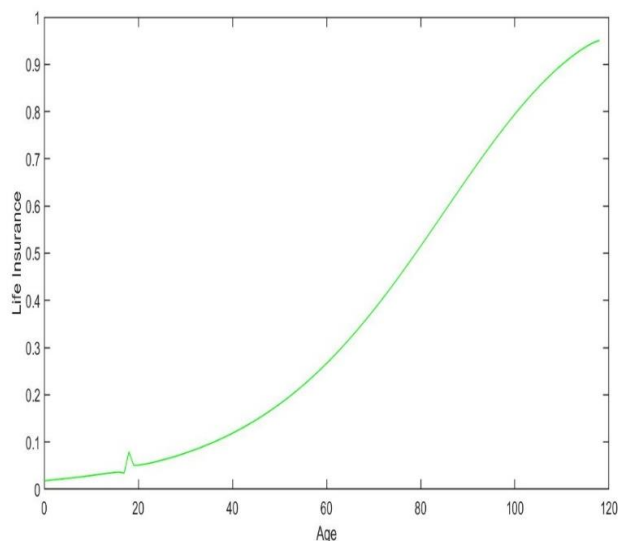


Figure 5 Female HP 3 Life Insurance Annuity

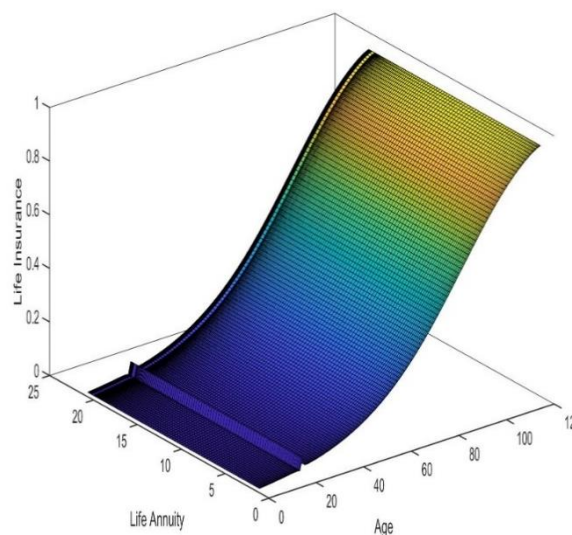


Figure 6 Female HP 3 Surface for Life Insurance and Life

Discussion of Results

The Heligman-Pollard's third law for the male and female annuity were represented in Table 1 with their corresponding curves shown in Figures 1 and 4 for both sexes. The kinks on the annuity curves were due to the mortality shocks around age 22-25 for the male for both sexes. Possibly, this may represent the interval where idiosyncratic risks of mortality or kinks had high impact on life annuity rates. The relationship among life annuity, life insurance and age is shown in the surface plot in Figures 3 and 6 for both male and female. For the Heligman-Pollard's third law, the life insurance benefits were represented in Table 1 with their corresponding curves shown in Figures 2 and 5 for both sexes while their surfaces were represented in Figures 3 and 6. Wavy kinks were also observed on the life insurance curves close to the middle of ages 20 and 25. The monotonic increasing behaviour of the benefit function was because mortality increase as age increase.

The Table 1 represents age-specific life tables for whole life annuity functions, whole life insurance functions and the associated net premiums for both sexes from ages 0 through 118. These functions constitute the basic numeric quantities in life contingencies and shows the joint impact of mortality and interest rate assumptions incorporated into the

governing benefits Table 1. The behaviour of the benefits table enhances detailed comparison of survival and death benefits present values over age and sex and it offers a holistic actuarial valuation technique over the whole life span.

The whole life annuities show the expected decreasing trend with age for both sexes. At age zero, the life annuity is maximal demonstrating the long expected random lifetime T_x and hence the expanded flow of annuity payments. When the age advances, the annuity falls steadily and this reduction is actuarially implicit. With increasing age, the expected number of future payments falls as a result of increasing mortality rate and the present value of distant payments is further reduced by actuarial discounting. The trend is smooth and monotonic within most age ranges to confirm consistency of the mortality assumptions. At extreme ages of Table 1, the life annuity tends to 1 showing that survivorship after additional year will not likely increase and the present value is governed by the probability of surviving one single period. The life annuity and life insurance columns are the basic commutation functions under defined mortality assumptions and interest rate assumption. The premium column is the annual net premium rate for

whole life insurance estimated from the equivalence relation $P_x = \frac{A_x}{\ddot{a}_x}$. This expression

is consistent with the values estimated in the benefits table 1.

A comparative sex differential is obvious in the life annuities because in almost all the ages, female life annuities are greater than male life annuities. This is a direct actuarial consequence of female longevity advantage. Because female gender experience smaller mortality rates at most ages, their expected future lifetime is longer, leading to a higher expected present value of annuity payments. The size of this deviation widens within middle and older ages, when mortality rate differentials between sexes are mostly substantial. The progressive difference shows that the Table 1 governing the computations effectively accounts for gender-based mortality distinctions. At age 0, $\ddot{a}_0^{(M)} = 20.45609723$ while

$\ddot{a}_0^{(F)} = 20.62958174$. These annuities are substantial since the expected future lifetime is maximum at age zero. The life annuity factor is the actuarial present value of a life-contingent annuity-due of 1 per year. Because young lives have long expected remaining lifetimes, the present value of future payments is high. In both childhood and adolescence, both male and female annuities fall gradually. The female life annuities are consistently bigger than male annuities and hence $\ddot{a}_x^{(F)} > \ddot{a}_x^{(M)}$. At almost all ages except extreme tail irregularities, females have higher annuity values. The bigger female annuities demonstrate smaller mortality rates and bigger life expectancy. Because females are expected to earn annuity payments for an extended period, the present value of their life annuity is bigger at each age.

Within ages 30 and 60, $\ddot{a}_x^{(M)}$ declines from approximately 18.64436509 at age 30 to 13.51785382 at age 60 while between ages 30 and 60, $\ddot{a}_x^{(F)}$ declines from approximately 19.39026861 at age 30 to 15.40211862 at age 60. The decline accelerates as age as a result of (i) reducing remaining lifetime, (ii) increasing mortality rates and (iii) discounting

reducing the weight of distant payments. This repeated gender gap is a consequence of systemic longevity advantage for females. At age 80, $\ddot{a}_{80}^{(M)} = 7.88825737$ while

$\ddot{a}_{80}^{(F)} = 10.18300009$, at age 100,

$\ddot{a}_{100}^{(M)} = 3.18939277$ while

$\ddot{a}_{100}^{(F)} = 4.31794033$. At extreme ages,

$\ddot{a}_{118}^{(M)} = 1.38338916$ while

$\ddot{a}_{118}^{(F)} = 1.03645878$. Although within these ages, the present value is governed by the probability of surviving a further year, the life annuity tends to 1 as the insured may not survive further one year.

The whole life insurance functions increase with age. Within young ages, the life insurance is low since the insured may not die in the short term and any potential benefit payment is massively discounted as a result of long deferred time before the expected death. As age advances, mortality rate increases and the expected period to payment shortens, reducing the discounting effect. As a result, the present value of the death benefit increases progressively. At very old ages, the insurance function tends to 1 showing the near certainty of death in a small-time interval and minimal effect of interest discounting. This monotonic upward behaviour is numerically consistent with classical actuarial life contingency principle. The gender differential is also obvious in the insurance functions as almost all ages, male insurance is higher than female insurance functional values. This is the corollary of the annuity relationship. Because males have bigger mortality rates, the actuarial present value of death benefits is higher for males at a defined age. The differential is obvious in mid-ages when male mortality is markedly bigger. This emphasises the mortality-driven characteristics of the mortality assumption and confirms actuarial coherence between the annuity and insurance columns. The insurance function is the expected present value of 1 unit whole life insurance benefit payable at death.

132 3e 0, $A_0^{(M)} = 0.02590013$ while $A_0^{(F)} = 0.01763896$ defining small insurance rates at young ages because (i) mortality rate is small (ii) death benefits are severely discounted and expected far into the future. Males show consistently bigger insurance than females as a result of his bigger mortality rates and hence, $A_x^{(M)} > A_x^{(F)}$. Males have higher insurance present values due to higher mortality. At age 40, $A_{40}^{(M)} = 0.16767423$ while $A_{40}^{(F)} = 0.11867245$. At age 60, $A_{60}^{(M)} = 0.35629268$ while $A_{60}^{(F)} = 0.26656578$. The insurance function grows steadily with age as a result of (i) probability of death increasing (ii) time to expected payment shortens and less discounting. The following increasing trends are noted. At age 80, $A_{80}^{(M)} = 0.62436870$ while $A_{80}^{(F)} = 0.39523$ At age 100, $A_{100}^{(M)} = 0.84812415$ while $A_{100}^{(F)} = 0.79438379$. As age tends to the limiting age of Table 1, $A_x \rightarrow 1$. This shows near certainty of death in a short time horizon and minimal discounting effect. The net premium column reflects premiums estimated under the equivalence principle, whereby the expected present value of premiums equals the expected present value of benefits. The premium pattern increases sharply with age. At young ages, premiums are extremely small because intensity of mortality risk is small. The premium column defines the actuarially fair annual net premium per unit sum assured. At age 0, $P_0^{(M)} = 0.00126613$ while $P_0^{(F)} = 0.00085503$. The premium is very small because (i) probability of death is very low (ii) expected claim cost is minimal (iii) payments are spread over a long premium-paying period. The male premium is higher than the female premium because of higher mortality rate intensities and hence $P_x^{(M)} > P_x^{(F)}$. Therefore, this reflects pure mortality-driven pricing under the equivalence

principle. The premium monotonically increases as age. At age 40, $P_{40}^{(M)} = 0.00959298$ while $P_{40}^{(F)} = 0.00641200$, while at age 60, $P_{60}^{(M)} = 0.02635719$ while $P_{60}^{(F)} = 0.01730709$. At age 100, $P_{100}^{(M)} = 0.26592026$ while $P_{100}^{(F)} = 0.18397285$. This increase is actuarially consistent the equivalence principle. As age increases, (i) A_x increases because of higher probability of death (ii) $\ddot{a}_x$ decreases as a result of shorter premium-paying period. Hence the ratio in (116) grows rapidly at advanced ages.

Conclusion

The benefit table, shows viable internal consistency as a result of monotonic decrease of annuity values, monotonic increase of insurance values, rapid acceleration of premiums at older ages and convergence behavior at limiting age. These patterns indicate a well-behaved mortality model, proper integration of interest discounting and actuarially coherent net premium calculations. The benefit Table 1 provides empirical confirmation of classical actuarial relationships (i) duality between annuity and insurance functions (ii) mortality-driven gender differentials (iii) increasing convexity of insurance function and (iv) non-linear escalation of net premiums at advanced ages.

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